On the Analysis of The Displacements in a Finite Isotropic Cracked Wedge Under Anti Plane Shear Deformation

[1]Adimoha,B.A.,[1]Nnadi,J.N.,[2]Emenogu,G.N.&[1*]Osu,B.O.
[1]Department of Mathematics,AbiaStateUniversity,Uturu,Nigeria.[2]Department of Mathematics, Michael Okpara University of Agriculture Umudike Nigeria.
[*]Corresponding Author: Osu.bright@abiastateuniversity.edu.ng
(https://orcid.org/0000-0003-2463-430X)
DOI: 10.56201/ijcsmt.v10.no6.2024.pg59.72

Abstract
A finite isotropic cracked wedge under anti plane shear deformation with a crack along $\theta = 0, 0 \leq r \leq c$ and a concentrated traction load along its radial edges is analyzed with particular consideration given to the displacement fields everywhere in the wedge. The problem is formulated using the Finite Mellin transform of the second kind, and the solution accompanied by employing the Wiener Hopf technique to obtain a closed form solution for the displacement everywhere in the wedge. It is found that at the region containing the crack, there is no continuity of the displacement fields, while continuity of the displacement fields exist in other regions of the wedge material.

Keywords: *Finite Mellin transform, Wiener-Holf ,Anti plane shear; Isotropic wegde.*

INTRODUCTION

The stress analysis in finite wedges has been the subject of numerous researches. Probably, the major interest in the wedge geometry can be accorded to the fact that half plane and edge cracks are special cases of wedge problems. Wedges are important geometries in the mathematical theory of elasticity, not only because of their applications in engineering problems but also, in a special case, a wedge can resemble other important geometries. Lap joints that have extensive applications in industries are examples of wedge applications. On the other hand, the problem regarding wedge-shaped geometry can be reduced to that of a quarter plane, a half plane, a plane, and an edged-cracked plane or a shaft, which are of important applications in the theory of elasticity and fracture mechanics.

The problem of cracked isotropic wedges under anti plane shear deformation has been under consideration for decades. Analytical approach to the problem, under anti plane shear loading was presented by Erdogan and Gupta (1972). The anti plane shear deformation of a bimaterial wedge with finite radius was studied by Shahani (2007) for various boundary conditions. The solution of the governing differential equation was accompanied by the means of finite Mellin transform.The closed-form solution for the displacement and the stress fields in the entire domain were obtained. Shahani and Adibnazari (2000) studied the problem of anti plane shear deformation of perfectly bonded wedges as well as bonded wedges having infinite radii with an interfacial crack by means of the Mellin transform. Ma and Hour (1990) studied the anti plane shear deformation of dissimilar anisotropic wedges by adopting finite Mellin transform. Choi and Earmme (1990) investgated the problem of the kinked crack in anti plane shear with emphasis on the order of performing the two limit processes when the kinked length goes to zero. They adopted the method of Mellin transform to formulate the problem, and solved the boundary value problem by the Wiener-Hopf technique, and obtained a closed form solution.

In this article, emphasis is given to the displacement fields everywhere in a finite isotropic cracked wedge under anti plane shear deformation with a traction-traction concentrated load prescribed on the radial edges of the wedge.

PROBLEM FORMULATION

A finite isotropic wedge with equal apex angles $\boldsymbol{\phi}$ is considered (see appendix for diagrams). A crack which lies on the line $\theta = 0 , 0 \leq r \leq c$ exists on the wedge. The condition of anti plane shear deformation is imposed on the wedge by the application of concentrated loads of magnitude T along $\theta = \boldsymbol{\phi}$ and $\theta = -\boldsymbol{\phi}$ at a distance h from the origin. Traction boundary conditions are assumed to act on the radial edges of the wedge, however, the faces of the crack is traction free. In such conditions, the only non-zero displacement component is the out-of-plane component, $W(r,\theta)$ which is in the z-direction and which is a function of the in-plane coordinates $r \text{ and } \theta$.

The constitutive relationship for isotropic materials undergoing anti plane deformation are

$$\sigma_{\theta z}(r,\theta) = \frac{\mu}{r}\frac{\partial W(r,\theta)}{\partial \theta} , \sigma_{rz}(r,\theta) = \mu\frac{\partial W(r,\theta)}{\partial r} \quad (1)$$

With the loading and known equations of elasticity, the boundary value problem that governs the wedge problem is given in terms of displacement as

$$\left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) W(r,\theta) = 0 , 0 \leq r \leq c , -\boldsymbol{\phi} \leq \theta \leq \boldsymbol{\phi} \quad (2)$$

The boundary conditions are:

On the radial surfaces

$$\sigma_{\theta z}(r,\phi) = T\delta(r-h) , \ 0 \leq r \leq a$$

$$\sigma_{rz}(r,-\phi) = T\delta(r-h) , 0 \leq r \leq a \quad (3)$$

On the circular arc

$$\sigma_{rz}(a,\theta) = 0 , -\boldsymbol{\phi} \leq \theta \leq \boldsymbol{\phi} \quad (4)$$

On the crack surface

$$W(r,0^+) \neq W(r,0^-) , 0 \leq r \leq c \quad (5)$$

$$\sigma_{\theta z}(r,0 \quad) = 0 , 0 \leq r \leq c$$

The continuity conditions are

$$W(r,0^+) = W(r,0^-) , c < r \leq a \quad (6)$$

$$\sigma_{\theta z}(r,0^+) = \sigma_{\theta z}(r,0^-) , c < r \leq a$$

The asymptotic behavior of the displacement is

$$W(r,\theta) = O\left(r^{1-\lambda}\right), 0 < \lambda < 1, as\, r \to 0 \qquad (7)$$

The finite Mellin transform of the second kind is defined as

$$M_2[W(r,\theta), s] = \widehat{W}(s,\theta) = \int_0^a \left(\frac{a^{2s}}{r^{s+1}} + r^{s-1}\right) W(r,\theta)\, dr \qquad (8)$$

Where s is a complex transform parameter .Inversion of this transform is defined by

$$M^{-1}\left[\widehat{W}(s,\theta), r\right] = W(r,\theta) = \frac{(-1)^j}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \widehat{W}\,(s,\theta) r^{-s} ds \qquad (9)$$

Application of (8) to the Laplace equation (2) in conjunction with the Leibnitz rule for differentiating the integral and integration by parts yields

$$\frac{\partial^2 \widehat{W}}{\partial \theta^2}(s,\theta) + s^2 \widehat{W}(\mathrm{s},\theta) = -2a^{s+1}\frac{\partial W(a,\theta)}{\partial r} \qquad (10)$$

Applying the boundary condition (4) on (10) with the aid of first of (1) leads to

$$\left(\frac{d^2}{d\theta^2} + s^2\right)\widehat{W}(s,\theta) = 0 \qquad (11)$$

Provided that

$$\lim_{r\to 0}\left[(a^{2s}r^{-s} + r^s) r \frac{\partial W(a,\theta)}{\partial r} + s(a^{2s}r^{-s} - r^s) W(a,\theta)\right] = 0 \qquad (12)$$

Referring to (12) in view of (7) leads to

$$\lim_{r\to 0}\left[(a^{2s}r^{-s} + r^s) r^{1-\lambda} + s(a^{2s}r^{-s} - r^s) r^{1-\lambda}\right] =$$

$$\operatorname{Lim}_{r\to 0}\left[(a^{2s}r^{-2s} + 1) r^{s+1-\lambda} + s(a^{2s}r^{-2s} - 1) r^{s+1-\lambda}\right] =$$

$$\lim_{r\to 0}[(1+s)a^{2s}r^{-2s} + 1 - s] r^{s+1-\lambda}$$

But,$r < a\ implies\ \frac{a}{r} > 1\ or\ \left(\frac{a}{r}\right)^{2s} > 1$

Hence

$$(1+s)a^{2s}r^{-2s} + (1-s) > (1+s) + (1-s) = 2$$

Implies that

$$r^{s+1-\lambda} \to 0\ as\ r \to 0$$

if

$$Res\ 1 - \lambda > 0\ \ or\ Res > \lambda - 1 \qquad (13)$$

where Re means real part of.

Considerthesolutionof(11)beoftheform

$$\widehat{W}(s,\theta) = A_1(s)cos\theta s + A_2(s)sin\theta s\,, 0 \leq \theta \leq \boldsymbol{\phi} \quad (14)$$

$$\widehat{W}(s,\theta) = B_1(s)cos\theta s + B_2(s)sin\theta s\,, -\boldsymbol{\phi} \leq \theta \leq 0$$

WIENER-HOPF METHOD OF SOLUTION

By employing the continuity conditions and the Wiener-Hopf technique, Noble (1958), for the determination of the coefficients $A_i(s)$ and $B_i(s)$, $i = 1,2$ we obtain

$$\frac{\mu}{2}E_+(s) = N(s)\left[F_-(s) - \frac{TK(s)\left(\frac{h}{c}\right)^s}{cos\phi s}\right] \quad (15)$$

where

$$N(s) = \frac{N_-(s)}{N_+(s)} \;\; = \frac{cos\phi s}{ssin\phi s} \quad (16)$$

$$F_-(s) = \int_1^{\frac{a}{c}}\left[\left(\frac{a}{c}\right)^{2s}\frac{1}{\rho} + \rho^s\right]\sigma_{\theta z}(c\rho,0)d\rho \quad (17)$$

$$E_+(s) = \int_0^1\left(\frac{(ac^{-1})^{2s}}{\rho^{s+1}}\rho^{s-1}\right)[W(c\rho,0^+) - W(c\rho,0^-)]d\rho \quad (18)$$

The coefficients in (14) can now be written in terms of (17) as

$$A_1(s) = \frac{c^s}{\mu s}F_-(s)\frac{cos\phi s}{sin\phi s} - \frac{T}{\mu s}\frac{K(s)h^s}{sin\phi s} \quad (19)$$

$$A_2(s) = \frac{c^s}{\mu s}F_-(s)$$

$$B_1(s) = -\frac{c^s}{\mu s}F_-(s)\frac{cos\phi s}{sin\phi s} + \frac{T}{\mu s}\frac{K(s)h^s}{sin\phi s}$$

$$B_2(s) = \frac{c^s}{\mu s}F_-(s)$$

It can be seen from (7) that $F_-(s)$ is analytic for $Res < \frac{1}{2}$, so is a left half hand function. The function $E_+(s)$ is analytic in the strip$Res > \lambda - 1$, so is a right half hand function.

Equation (15) will be solved to determine $F_-(s)$ and $E_+(s)$ based on Wiener-Hopf technique. The function $N(s)$ in (15) is decomposed as (16).

To find $N_-(s)$ and $N_+(s)$, we use the infinite product expansion of $cos\boldsymbol{\phi}s$ and $sin\boldsymbol{\phi}s$ (Korn and Korn (1968))given as

$$cos\boldsymbol{\phi}s = \prod_{n=1}^{\infty}\left(1 - \left[\frac{2\phi s}{(2n-1)\pi}\right]^2\right) \quad (20)$$

and

$$sin\boldsymbol{\phi}s = \boldsymbol{\phi}s\prod_{n=1}^{\infty}\left[1-\left(\frac{\phi s}{n\pi}\right)^2\right] \tag{21}$$

To get

$$N_+(s) = \boldsymbol{\phi}s^2\frac{\Gamma\left(\frac{2\phi s}{\pi}+1\right)}{\left[\Gamma\left(\frac{\phi s}{\pi}+1\right)\right]^2}e^{\boldsymbol{\psi}\boldsymbol{s}} \tag{22}$$

which is never zero, except if s=0.

On the other hand,

$$N_-(s) = \frac{\Gamma\left(1-\frac{2\phi s}{\pi}\right)}{\left[\Gamma\left(1-\frac{\phi s}{\pi}\right)\right]^2}e^{\psi s} \tag{23}$$

where $e^{\psi s}$is introduced to make the expressions in (22) and (23) to have algebraic behavior for large s. The algebraic behavior of the gamma function as$|s| \to \infty$ is obtained by analyzing

$$\Gamma(\beta w + 1) = \sqrt{2\pi\beta w}(\beta w)^{\beta w}e^{-\beta w\beta} \tag{24}$$

And applying (22) and noting that $\beta = \frac{\phi}{\pi}$ $and\ w = s, we$ obtain the following results:

$$\psi = -\frac{2\phi}{\pi}\ln 2 \tag{25}$$

$$N_+(0) = 0 \tag{26}$$

$$N_+(s) = \phi^{\frac{1}{2}}s^{\frac{3}{2}}\ as\ |s| \to \infty \tag{27}$$

Now,substituting (16) into (15) we obtain

$$\frac{\mu}{2}E_+(s)N_+(s) = N_-(s)F_-(s) - TN_-(s)\frac{K(s)\left(\frac{h}{c}\right)^s}{cos\phi s} \tag{28}$$

$$= N_-(s)F_-(s) - TL(s)$$

where

$$L(s) = \frac{K(s)\left(\frac{h}{c}\right)^s N_-(s)}{cos\phi s} \tag{29}$$

Now, we decompose L(s) as

$$L(s) = T[L_+(s) + L_-(s)] \tag{30}$$

Then, using Mittag-Leffler's expansion theorem of $sec\boldsymbol{\phi}s$ given in(Spiegel,et'al,2009) to get

$$L_+(s) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{\frac{\phi s}{\pi}+\alpha_n}K\left(-\frac{\pi}{\phi}\alpha_n\right)\left(\frac{h}{c}\right)^s N_-\left(-\frac{\pi}{\phi}\alpha_n\right) \quad (31)$$

$$L_-(s) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{\frac{\phi s}{\pi}-\alpha_n}K(s)\left(\frac{h}{c}\right)^s N_-(s) \quad (32)$$

where

$$\alpha_n = \frac{2n-1}{2}$$

Because the function analytic in the left half plane are equal to the function analytic in the right half plane, each function is an analytic continuation of the other with the fundamental strip as the strip of equality. Therefore, each side is bounded and analytic in the entire s-plane. By Liouville's theorem (Spiegel,2009), they are equal to a constant. In other words,

$$\frac{\mu}{2}E_+(s)N_+(s) + TL_+(s) = N_-(s)F_-(s) - TL_-(s) = H_0 \quad (33)$$

Since (33) is true for all s, it must be true for s=0.

The constant H_0 can be determined from the behavior of $N_-(s)F_-(s)\ and\ L_-(s)$ at s=0, and it is given by

$$H_0 = TL_+(0) \quad (34)$$

where

$$E_+(0)N_+(0) + TL_+(0) = H_0 \quad (35)$$

$$F_-(0)N_-(0) - TL_-(0) = H_0 \quad (36)$$

but

$$E_+(0) \neq 0$$

$$F_-(0) \neq 0$$

so, (33) yields

$$E_+(s) = \frac{2T}{\mu}\frac{[L_+(0)-L_+(s)]}{N_+(s)} \quad (37)$$

and

$$F_-(s) = T\frac{[L_+(0)+L_-(s)]}{N_-(s)} \quad (38)$$

then, (33) now becomes

$$-\frac{\mu}{2}E_+(s)N_+(s) + F_-(s)N_-(s) = T[L_+(s) + L_-(s)] = TL(s) \quad (39)$$

Use made of (16), (29) and (33), $F_-(s)$ can now be written in terms of $E_+(s)$ as

$$F_{-}(s) = \frac{\mu}{2} E_{+}(s) \frac{s sin\phi s}{cos\phi s} + \frac{TK(s)\left(\frac{h}{c}\right)^{s}}{cos\phi s} \tag{40}$$

Consequently, the coefficients in (14) can then be written in terms of $E_{+}(s)$ through (19) and (40) as

$$A_1(s) = \frac{c^s}{2} E_{+}(s) \tag{41}$$

$$A_2(s) = \frac{c^s}{2} E_{+}(s) \frac{sin\phi s}{cos\phi s} + \frac{T}{\mu s} \frac{K(s)h^s}{cos\phi s}$$

$$B_1(s) = -\frac{c^s}{2} E_{+}(s)$$

$$B_2(s) = \frac{c^s}{2} E_{+}(s) \frac{sin\phi s}{cos\phi s} + \frac{T}{\mu s} \frac{K(s)h^s}{cos\phi s}$$

Then the transformed displacement sought for in (14) is derived for $0 \leq \theta \leq \boldsymbol{\phi}$ in terms of $F_{-}(s)$ using

(19) and (40) as

$$\widehat{W}(s,\theta) = \frac{1}{\mu s}\left[F_{-}(s) \frac{cos(\phi-\theta)s}{sin\phi s} - TK(s) \frac{cos\theta s\left(\frac{h}{c}\right)^{s}}{sin\phi s}\right] c^s \tag{42}$$

And for $-\boldsymbol{\phi} \leq \theta \leq 0$, we get

$$\widehat{W}(s,\theta) = \frac{1}{\mu s}\left[-F_{-}(s) \frac{cos(\phi+\theta)s}{sin\phi s} + TK(s) \frac{cos\theta s\left(\frac{h}{c}\right)^{s}}{sin\phi s}\right] c^s \tag{43}$$

Also, in terms of $E_{+}(s)$, the transformed displacement is derived using (14), (41) for $0 \leq \theta \leq \boldsymbol{\phi}$ as

$$\widehat{W}(s,\theta) = \left[\frac{E_{+}(s)}{2} \frac{cos(\phi-\theta)s}{cos\phi s} + \frac{T}{\mu} \frac{K(s)\left(\frac{h}{c}\right)^{s} sin\theta s}{s cos\phi s}\right] c^s \tag{44}$$

for $-\boldsymbol{\phi} \leq \theta \leq 0$, we have

$$\widehat{W}(s,\theta) = -\left[\frac{E_{+}(s)}{2} \frac{cos(\phi+\theta)s}{cos\phi s} - \frac{T}{\mu} \frac{K(s)\left(\frac{h}{c}\right)^{s} sin\theta s}{s cos\phi s}\right] c^s \tag{45}$$

RESULTS AND DISCUSSIONS

The displacement fields everywhere in the finite wedge are obtained through the inversion formula defined by (9). Two regions emerge (see appendix for diagram), each made up of two sub regions for estimation of the displacement fields in accordance with the conditions that ensure associated series converge. One region is formed by two other sub regions that occur when $0 \leq \theta \leq \boldsymbol{\phi}$, $0 \leq r \leq c$ and the powers in $\left(\frac{r}{c}\right)^{-s}$, obtained from the application of residue theorem, are required to possess positive values. For this case, s should have negative values for corresponding residue appropriate for the domain of $F_{-}(s)$. The other sub regions also originates from consideration of residue appropriate for the domain of $F_{-}(s)$ when $-\boldsymbol{\phi} \leq \theta \leq 0$ and $0 \leq r \leq c$. The two series converge as $r \to 0$.

The second region is also made up of two sub regions, but associated with positive residues appropriate for the domain of $E_{+}(s)$.The two sub regions originate from consideration for the ranges $0 \leq \theta \leq \boldsymbol{\phi}$, $-\boldsymbol{\phi} \leq \theta \leq 0$ and $c < r \leq a$. It is required that the power of $\left(\frac{r}{c}\right)^{-s}$, obtained from the application of residue theorem, be positive so that the series converges as $r \to \infty$.

The displacements everywhere in the wedge is then derived as following:

FOR THE REGION CONTAINING THE CRACK

$(0 \leq r \leq c\ , -\boldsymbol{\phi} \leq \theta \leq \boldsymbol{\phi})$.

Using (9) , (42) , (43)for the regions $0 \leq \theta \leq \boldsymbol{\phi}$, $and - \boldsymbol{\phi} \leq \theta \leq 0$,

$0 \leq r \leq c$ for both respectively yields

$$W(r,\theta) = \frac{1}{\mu s}\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{F_{-}(s)\frac{\cos(\phi-\theta)s}{\sin\phi s} - T\frac{K(s)\left(\frac{h}{c}\right)^{s}\cos\theta s}{\sin\phi s}\right\}\right]\left(\frac{r}{c}\right)^{-s} ds \tag{46}$$

and

$$W(r,\theta) = \frac{1}{\mu s}\left[\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{-F_{-}(s)\frac{\cos(\phi+\theta)s}{\sin\phi s} + T\frac{K(s)\left(\frac{h}{c}\right)^{s}\cos\theta s}{\sin\phi s}\right\}\right]\left(\frac{r}{c}\right)^{-s} ds \tag{47}$$

The Bromwich integrals in (46) and (47) can be evaluated by Cauchy's residue theorem in accordance with Jordan's Lemma.

The integrands has a pole of order 2 at s=0 and simple poles at $s_n = \pm\frac{n\pi}{\phi}$, $n = 1,2,3, \ldots$We will be concerned with simple poles $s_n = -\frac{n\pi}{\phi}$, $n = 1,2,3, \ldots$for a reason earlier given. Then, we obtain the sum of residues for the displacements in their dominant term when n=1 as $r \to 0$ for both regions as

$$W(r,\theta) = \frac{1}{\mu\pi}\left[F_{-}\left(-\frac{\pi}{\phi}\right) + TK\left(-\frac{\pi}{\phi}\right)\left(\frac{c}{h}\right)^{\frac{\pi}{\phi}}\right]\cos\frac{\pi\theta}{\phi}\left(\frac{r}{c}\right)^{\frac{\pi}{\phi}} , r < c \tag{48}$$

and

$$W(r,\theta) = -\frac{1}{\mu\pi}\left[F_{-}\left(-\frac{\pi}{\phi}\right) + TK\left(-\frac{\pi}{\phi}\right)\left(\frac{c}{h}\right)^{\frac{\pi}{\phi}}\right]\cos\frac{\pi\theta}{\phi}\left(\frac{r}{c}\right)^{\frac{\pi}{\phi}} , r < c \tag{49}$$

DISPLACEMENT FOR THE REGION AHEAD OF THE CRACK

$(-\boldsymbol{\phi} \leq \theta \leq \boldsymbol{\phi}, c < r \leq a)$.

The displacement for these regions, $0 \leq \theta \leq \boldsymbol{\phi}$ and $-\boldsymbol{\phi} \leq \theta \leq 0$, both for $c < r \leq a, r \to \infty$ or

$\frac{1}{r} \to 0$ is evaluated with the help of (9) , (44) and (45) respectively as

$$W(r,\theta) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\frac{E_+(s)}{2}cos(\phi-\theta)s + \frac{T}{\mu}K(s)\left(\frac{h}{c}\right)^S \frac{sin\theta s}{s}\right\}\frac{1}{cos\phi s}\left(\frac{r}{c}\right)^{-s} ds \tag{50}$$

and

$$W(r,\theta) = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\frac{E_+(s)}{2}cos(\phi+\theta)s - \frac{T}{\mu}K(s)\left(\frac{h}{c}\right)^S \frac{sin\theta s}{s}\right\}\frac{1}{cos\phi s}\left(\frac{r}{c}\right)^{-s} ds \tag{51}$$

The integrands in (50) and (51) are to be evaluated by the residue method. The integrands has simple poles at $s_n = \pm\frac{(2n-1)}{2}\frac{\pi}{\phi}$, $n = 1,2,3,\ldots$ We will be concerned with simple poles at $s_n = \frac{(2n-1)}{2}\frac{\pi}{\phi}$.

Using the same approach as in the region above, we now obtain the sum of residues for the displacement fields in their dominant term when $n = 1$, as $r \to \infty$, $\frac{1}{r} \to 0$ for both regions as

$$W(r,\theta) = -\frac{1}{\phi}\left\{\frac{E_+\left(\frac{\pi}{2\phi}\right)}{2\phi} + \frac{2T\phi}{\mu\pi}K\left(\frac{\pi}{2\phi}\right)\left(\frac{h}{c}\right)^{\frac{\pi}{2\phi}}\right\} sin\frac{\pi\theta}{2\phi}\left(\frac{c}{r}\right)^{\frac{\pi}{2\phi}}, r > c \tag{52}$$

and

$$W(r,\theta) = -\frac{1}{\phi}\left\{\frac{E_+\left(\frac{\pi}{2\phi}\right)}{2\phi} + \frac{2T\phi}{\mu\pi}K\left(\frac{\pi}{2\phi}\right)\left(\frac{h}{c}\right)^{\frac{\pi}{2\phi}}\right\} sin\frac{\pi\theta}{2\phi}\left(\frac{c}{r}\right)^{\frac{\pi}{2\phi}}, r > c \tag{53}$$

CRACK TIP DISPLACEMENT

In order to obtain the crack tip displacement, a local polar coordinate (ρ, ψ) is introduced with the origin at the wedge apex (see appendix for diagram).The main property of the coordinate in this analysis is that

$$\theta \approx 0, \boldsymbol{\psi} \approx 0, \theta = \boldsymbol{\psi}, r \to c \text{ as } \rho \to 0 \tag{54}$$

Using the vector triangle rule, we obtain

$$\frac{r}{c} = 1 + \frac{\rho}{c}cos\boldsymbol{\psi} + O(\rho^2) \text{ as } \rho \to 0 \tag{55}$$

then, the crack tip displacement is studied through (44) and (45) for the regions $0 \leq \theta \leq \boldsymbol{\phi}$,

$-\boldsymbol{\phi} \leq \theta \leq 0$,both for $c < r \leq a$ respectively in terms of the local coordinate(ρ, ψ). Then, we set

$$W(\rho,\psi) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{E_+(s)}{2}\left(\frac{r}{c}\right)^{-s} ds\,, \lambda - 1 < \sigma < \frac{1}{2} \qquad (56)$$

$$W(\rho,\psi) = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{E_+(s)}{2}\left(\frac{r}{c}\right)^{-s} ds\,, \lambda < 1 < \sigma < \frac{1}{2} \qquad (57)$$

and $\theta \to \boldsymbol{\psi}\,, r \to \rho \; and \; \theta = 0\,, \rho \to 0$.

Using the method of Choi and Earmme (1990), the integral in (56) and (57) is computed by setting $\sigma = 0$ and splitting the symmetric interval $(-i\infty\,, i\infty)$ into three such that

$$(-i\infty, i\infty) = \left(-i\infty, -i\left(\frac{\rho}{c}\right)^{\beta}\right) \cup \left(-i\left(\frac{\rho}{c}\right)^{\beta}, i\left(\frac{\rho}{c}\right)^{\beta}\right) \cup \left(i\left(\frac{\rho}{c}\right)^{\beta}, i\infty\right) \qquad (58)$$

Then, using (37) and the approximation in (56) in conjunction with entry 3.381(3) of Gradshyteyn and Ryzik (1965), we obtain the crack tip displacements for$0 \leq \theta \leq \boldsymbol{\phi}$ *and* $-\boldsymbol{\phi} \leq \theta \leq 0$,both for $c < r \leq c$ as

$$W(\rho,\psi) = \delta_0 + \frac{2TL_+(0)}{\mu\sqrt{\pi\psi}}\left(\frac{\rho}{c}\right)^{\frac{1}{2}} sin\frac{\psi}{2} \qquad (59)$$

and

$$W(\rho,\psi) = \varphi_0 + \frac{2TL_+(0)}{\mu\sqrt{\pi\psi}}\left(\frac{\rho}{c}\right)^{\frac{1}{2}} sin\frac{\psi}{2} \qquad (60)$$

where

$\delta_0 = \varphi_0$ is a constant defining a rigid body motion.

Now,the displacement fields obtained in (49) , (50) , (53) and (54) is now obtained in closed–form by substituting (37) and (38) into (49) , (50) , (53) and (54) respectively.

Then, for the region containing the crack ,$0 \leq \theta \leq \boldsymbol{\phi}\,, 0 \leq r \leq c$,we have

$$W(r,\theta) = \frac{T}{\mu}\left[\left[L_+(0) + L_-\left(-\frac{\pi}{\phi}\right)\right] + K\left(-\frac{\pi}{\phi}\right)\left(\frac{c}{h}\right)^{\frac{\pi}{\phi}}\right] cos\frac{\pi\theta}{\phi}\left(\frac{r}{c}\right)^{\frac{\pi}{\phi}} <, r < c \qquad (61)$$

for$-\boldsymbol{\phi} \leq \theta \leq 0\,, 0 \leq r \leq c$,weget

$$W(r,\theta) = -\frac{T}{\mu}\left[\left[L_+(0) + L_-\left(-\frac{\pi}{\phi}\right)\right] + K\left(-\frac{\pi}{\phi}\right)\left(\frac{c}{h}\right)^{\frac{\pi}{\phi}}\right] cos\frac{\pi\theta}{\phi}\left(\frac{r}{c}\right)^{\frac{\pi}{\phi}}, r < c \qquad (62)$$

FOR THE DISPLACEMENT IN THE REGION AHEAD OF THE CRACK

$(-\boldsymbol{\phi} \leq \theta \leq \boldsymbol{\phi}\,, c < r \leq a)$, we get for $0 \leq \theta \leq \boldsymbol{\phi}\,, c < r \leq a\,, r \to \infty\ or$

$\frac{1}{r} \to 0$

$$W(r,\theta) = -\frac{2T}{\mu\pi}\left[\left[\frac{L_+(0)-L_+\left(\frac{\pi}{2\phi}\right)}{2\phi}\right] + K\left(\frac{\pi}{2}\right)\left(\frac{h}{c}\right)^{\frac{\pi}{2\phi}}\right] sin\frac{\pi\theta}{2\phi}\left(\frac{c}{r}\right)^{\frac{\pi}{2\phi}}\ , r > c \qquad (63)$$

And for $-\boldsymbol{\phi} \leq \theta \leq 0\,,\ c < r \leq a\,,\ as\ r \to \infty\ or\ \frac{1}{r} \to 0$, we get

$$W(r,\theta) = -\frac{2T}{\mu\pi}\left[\left[\frac{L_+(0)-L_+\left(\frac{\pi}{2\phi}\right)}{2\phi}\right] + K\left(\frac{\pi}{2}\right)\left(\frac{h}{c}\right)^{\frac{\pi}{2\phi}}\right] sin\frac{\pi\theta}{2\phi}\left(\frac{c}{r}\right)^{\frac{\pi}{2\phi}}\ , r > c \qquad (64)$$

CONCLUSION

A finite isotropic cracked wedge with equal apex angleϕ,material constantμ and a crack which lies on the line$\theta = 0\,, 0 \leq r \leq c$ has been considered , with a particular attention given to displacements $W(r,\theta)$ everywhere in the cracked wedge.A closed-form of the solution is obtained in this study by the Wiener-Hopf technique .The result shows that:

i) at the region containing the crack,$0 \leq \theta \leq \boldsymbol{\phi}\,, 0 \leq r \leq c\ and - \boldsymbol{\phi} \leq \theta \leq 0\,, 0 \leq r \leq c, r \to 0$

$W(r,0^+) \neq -W(r,0^-)$

implies that there is no continuity of the displacement fields along the crack region

ii) at the region ahead of the crack$0 \leq \theta \leq \boldsymbol{\phi}\,, c < r \leq a\ and - \boldsymbol{\phi} \leq \theta \leq 0\,, c < r \leq a\,, r \to \infty$

$or\ \frac{1}{r} \to 0$

$W(r,0^+) = 0$

$W(r,0^-) = 0$

Implies that in the region ahead of the crack, there is continuity of the displacement fields. Therefore, all boundary conditions are satisfied.

REFERENCES

Choi,S.R & Earmme,Y.Y.(1990). Analysis Of A Kinked Cracked In Anti Plane Shear. *Mechanics Of Materials*, *9* , *195-204*.

Erdogan,F.& Gupta, G.D.(1972). Stress Near A Flat Inclusion In Bonded Dissilmilar Materials. *International Journal Of Solids And Structures* ,*8* , *533-547*.

Gradshyten ,I.M. & Ryzhik.(1965). Tables Of Integrals ,Series And Products, Academic Press, New York.

Karganovin,M.H.(2000). Analysis Of A Dissimilar Finite Wedge Under Anti Plane Deformation. *Mechanics Research Communication*, *27*(*1*), *109-116*.

Karganovin, M.H, Shahani, A.R , & Fariborz, S.J.(1997). Analysis Of An Isotropic Finite Wedge Under Anti Plane Deformation. *International Journal Of Solids And Structures,34,113-128*.

Korn,G.A & Korn,T.M .(1968). Mathematical Handbook For Scientists And Engineers, Dover Publications, Inc. Mineola ,New York.

Ma,C.C & Hour, B. (1990). Analysis Of Dissimilar Anisotropic Wedges Subjected To Anti Plane Deformation, *International Journal Of Solids And Structures*, *25,1295-1309*.

Noble,B. (1958). Methods Based On The Wiener-Hopf Technique For The Solutions Of Differentia lEquations, Pergamon Press, London.

Oucheterlony,F.(1980). Symmetric Cracking Of A Wedge By Transverse Displacement. *Journal Of Elasticity,10,109-116*.

Shahani,A.R. (2007). On the Anti Plane Shear Deformation Of Finite Wedges, *AppliedMathsModel,31,141-151*.

Shahani,A.R & Adibnazari, S. (2000). Analysis Of Bonded Finite Wedges With An Interfacial Crack Under Anti Plane Shear Loadin .*Journal Of Mathmatics And Engineering Science,223, 2212-2222.*

Spiegel, M.R. (2009). Complex Variable, Schaum Series, NewYork,174-175.

Tranter,C.J.(1956). Integral Transforms In Mathematical Physics,Willey,
NewYork.

APPENDIX

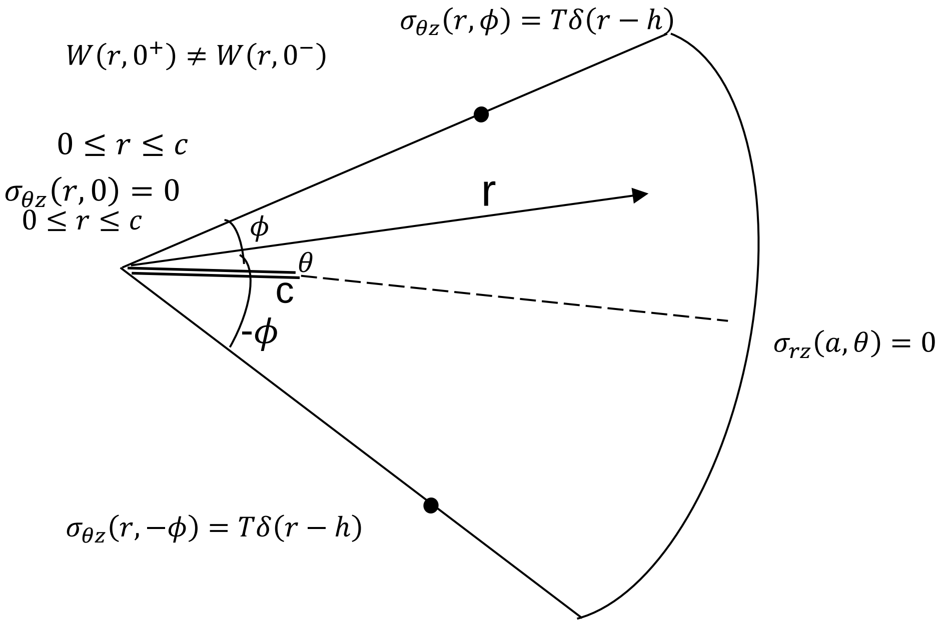

Figure I. GEOMETRY OF THE PROBLEM.

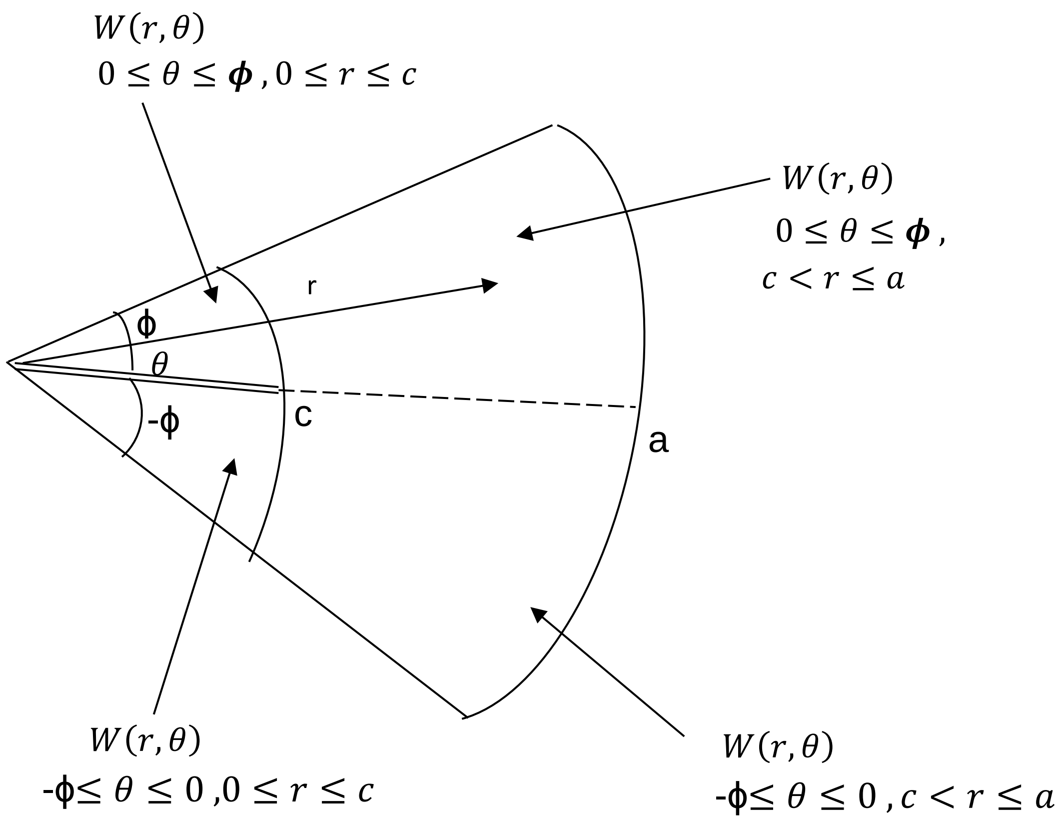

Figure II. DISTRIBUTION OF DISPLACEMENT FIELDS

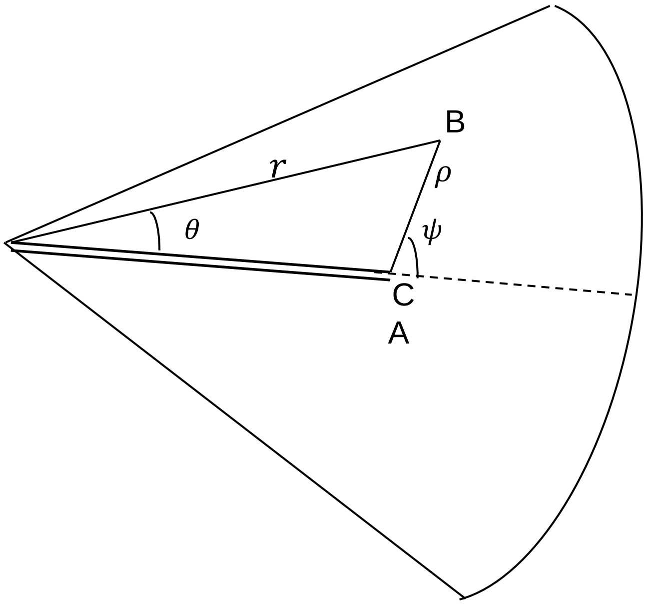

Figure III. LOCAL POLAR COORDINATE (ρ, ψ) AT THE CRACK TIP

FigureI.GEOMETRY OF THE PROBLEM.

FigureII. DISTRIBUTION OF DISPLACEMENT FIELDS